

# New Constructions for Partitionable Sets and Almost Partitionable Sets

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## Abstract

Partitionable sets (PS) and almost partitionable sets (APS) are two important types of combinatorial configurations in design theory. They have intimate connections with other combinatorial structures such as  $\mathbb{Z}$ -cyclic patterned starter whist tournaments, cyclic difference matrices, cyclic balanced sampling plans excluding contiguous units, disjoint difference families, and optical orthogonal codes. The existence problems of PS and APS remain far from settled as they demand stringent requirements. This paper focuses on the case of  $p \equiv 7 \pmod{8}$  being a prime and establishes new constructions for partitionable sets of order  $p^2$  and almost partitionable sets of order  $p$ . In particular, for  $p \equiv 7 \pmod{8}$ , this paper shows the existence of PS of order  $p^2$  for a large portion of primes  $p < 30000$ , the existence and asymptotic existence of APS under certain conditions, and also the existence of APS of order  $p$  for all primes  $p < 50000$  with 16 possible exceptions.

**MSC:** 05B05

**Keywords:** partitionable set, almost partitionable set, whist tournament, cyclotomic class

The paper entitled “New constructions for partitionable sets and almost partitionable sets” (in Chinese) will be published in Sci Sin Math, 2022 (52). Here we display several tables supporting some proofs of this paper.

Table I: Data for Theorem 2.1

$p$	$\sqrt{2}$	$\xi$	$s$	$a$	$p$	$\sqrt{2}$	$\xi$	$s$	$a$
3527	148050	12291678	3526	2114307	3583	1570413	11267475	3582	6201907
3607	3999301	3999302	1202	1106502	3623	1367950	1367951	3622	249301
3631	2953963	10230197	3630	4721811	3671	243008	243009	3670	1129971
3719	3514394	3514395	3718	5407291	3727	5241523	5241524	3726	3143798
3767	4531308	4531309	3766	5998240	3823	3962279	3962280	3822	6692247
3847	235454	14563954	3846	4625009	3863	7178771	7178772	3862	4386273
3919	706975	706976	3918	1626289	3943	3727764	3727765	3942	4655493
4007	7597533	7597534	4006	390049	4079	369234	16269006	4078	6744185
4127	8478636	8478637	4126	1452446	4231	194534	17706826	4230	8880464
4271	8316748	9924692	4270	7327537	4391	5241609	5241610	4390	7840483
4423	3871151	15691777	1474	2434117	4519	955146	19466214	4518	4642332
4567	6632075	14225413	4566	326622	4583	1006446	19997442	4582	2192057
4591	5650292	15426988	4590	4837192	4639	2729680	2729681	4638	8996555
4679	3235619	3235620	4678	10437298	4703	8207464	13910744	4702	5743561
4751	8521342	14050658	4750	8740690	4759	5744044	5744045	4758	3883290
4783	906287	21970801	4782	3889703	4799	2816134	20214266	4798	7110855
4871	8071879	15654761	4870	1689290	4919	2123926	2123927	4918	11172426
4967	2632032	2632033	4966	7850294					

Table II: Data for Lemma 3.4

$p$	$\sqrt{2}$	$\xi$	$\omega$	$t$	$(b_1 \ b_2 \ b_3 \ b_4)$	$p$	$\sqrt{2}$	$\xi$	$\omega$	$t$	$(b_1 \ b_2 \ b_3 \ b_4)$
751	113	114	3	5	(3 507 277 62)	1031	473	474	14	5	(14 1003 813 654)
1151	48	1102	17	5	(17 987 1076 891)	1279	229	1049	3	9	(3 49 838 1128)
2111	898	1212	7	5	(7 1107 1990 74)	2719	215	216	3	9	(3 1311 2586 778)
3191	1037	2153	11	5	(11 463 62 1772)	3391	1400	1990	3	5	(3 2304 784 25)
3911	234	3676	13	5	(13 1379 2513 529)	4111	1877	2233	12	5	(12 2879 1999 4081)
4159	1581	1582	3	9	(3 2821 3148 1595)	5231	1505	1506	7	5	(7 14 2725 2579)
5351	625	626	11	5	(11 2026 2469 4406)	5591	436	5154	11	5	(11 3657 111 4685)
6679	2879	3799	7	9	(7 1388 1063 5732)	6823	2025	2026	3	9	(3 2362 3101 2974)
7583	3493	3494	5	17	(5 7461 4192 5690)	7591	326	7264	6	33	(6 3101 5435 4112)
7687	124	125	6	21	(6 1511 1810 6621)	8191	128	8062	17	13	(17 1266 202 1974)
8231	1628	1629	11	5	(11 803 4116 5561)	8311	3843	4467	3	5	(3 4366 5079 6450)
9871	1689	8181	3	21	(3 393 8679 6259)	10271	798	9472	7	5	(7 7899 1297 4289)
10391	1360	1361	19	5	(19 6181 7645 7804)	10711	2960	7750	3	5	(3 9214 6057 2823)
11071	3990	7080	3	5	(3 2477 9934 2036)	12799	160	12638	13	9	(13 9655 5500 9379)
13751	5075	5076	11	5	(11 7491 4941 9631)	14591	2894	2895	11	5	(11 12638 5989 11254)
14831	2801	12029	11	5	(11 10800 5770 10210)	15551	1837	13713	7	5	(7 7606 2366 10493)
15671	7487	7488	13	5	(13 650 13758 5218)	16631	6502	10128	19	5	(19 14307 7026 95)
16831	2328	2329	6	9	(6 2538 15791 15008)	16927	184	16742	6	13	(6 9002 16240 1791)
17431	494	16936	3	5	(3 5074 16418 2495)	17599	8624	8974	6	21	(6 14812 10846 5700)
18311	6676	6677	13	5	(13 4312 7782 5762)	19471	2531	16939	11	5	(11 16524 3453 4917)
21031	3720	17310	12	5	(12 20960 16446 7163)	21191	1019	20171	7	5	(7 19975 9149 19175)
21751	1460	1461	3	5	(3 10819 4447 21684)	22159	7097	15061	6	9	(6 5687 3682 16741)
22871	10102	12768	7	5	(7 5146 605 1496)	23143	8809	8810	5	21	(5 5951 12183 8719)
23167	2120	2121	3	13	(3 15972 21590 7937)	23431	11513	11917	3	5	(3 1489 12952 21587)
23599	1633	21965	3	9	(3 16945 9028 19470)	23831	3216	3217	11	5	(11 19343 21777 13599)
23887	3913	3914	3	9	(3 19091 22030 13596)	26711	11659	11660	11	5	(11 21888 21707 26409)
27919	3890	3891	3	9	(3 13661 25511 14078)	28111	7485	20625	3	5	(3 1881 23120 27446)
28711	634	635	3	5	(3 27068 25629 4984)	28751	4073	24677	14	5	(14 198 473 27798)
28927	4979	23947	3	9	(3 20617 16992 26872)	29231	8251	8252	13	5	(13 28982 20236 28565)
29527	9287	9288	3	21	(3 26019 8673 19188)	32063	10911	21151	5	17	(5 10653 14500 7992)
33967	11739	11740	3	17	(3 2974 31447 3770)	35999	10072	10073	13	41	(13 10409 34649 24289)
36007	710	711	3	17	(3 29054 17233 20904)	36791	17217	19573	17	13	(17 33256 31081 4071)
38047	6984	6985	3	17	(3 17491 23613 35386)	38183	18833	19349	5	17	(5 17042 20835 35147)
39439	4360	4361	3	9	(3 110 9706 3374)	40087	4605	35481	3	9	(3 39338 6076 7739)
46639	6233	6234	6	9	(6 9884 40445 43592)	48247	8245	8246	3	17	(3 47717 22681 2054)
48647	9667	9668	5	13	(5 8149 8289 40046)						

Table III (1): Data for Lemma 3.5 with  $t = 3$ 

$p$	$\sqrt{2}$	$\xi$	$\omega$	$(x_1 \ x_2 \ b_i)$	$p$	$\sqrt{2}$	$\xi$	$\omega$	$(x_1 \ x_2 \ b_i)$
487	91	92	3	(13 38 378)	607	194	412	3	(5 10 461)
1327	255	1071	3	(53 88 797)	1399	650	748	13	(25 46 89)
1423	401	1021	3	(49 76 767)	1447	258	1188	3	(87 98 69)
1879	294	1584	6	(51 130 1817)	2311	68	69	3	(21 118 2041)
2647	300	2346	3	(37 198 1124)	2767	510	2256	3	(57 114 1576)
2887	76	77	5	(11 66 2429)	3079	847	848	6	(7 164 202)
3271	214	3056	3	(11 40 1010)	3511	559	560	7	(15 74 2181)
3559	1575	1576	3	(35 70 149)	3607	862	2744	5	(45 124 3529)
3967	63	3903	6	(69 256 3013)	4327	969	3357	3	(21 98 685)
4423	1026	3396	3	(19 48 3181)	4831	2222	2608	3	(19 130 1885)
4903	262	263	3	(15 198 4449)	4999	100	101	3	(39 112 3491)
5527	2358	3168	5	(43 204 3280)	5743	1743	3999	10	(11 284 1202)
5839	1225	4613	6	(37 448 4116)	6367	329	330	3	(13 354 5621)
6703	2658	4044	5	(3 144 1276)	6871	2318	2319	3	(49 244 2573)
7159	3118	4040	3	(19 280 4372)	7759	872	6886	3	(35 42 6038)
8599	3421	3422	3	(5 224 3078)	8863	1959	1960	3	(3 696 7841)
8887	3267	5619	3	(11 630 7090)	9127	4161	4162	3	(3 304 7522)
9151	3732	3733	3	(25 142 6506)	9343	2581	6761	5	(17 746 369)
9463	3632	3633	3	(11 236 4675)	9511	4484	4485	3	(3 498 8430)
9631	3715	3716	3	(57 716 7594)	9967	3038	3039	3	(21 670 4100)
10303	1079	9223	3	(7 74 4934)	10567	5037	5529	6	(15 216 485)
10663	1873	8789	3	(43 636 4640)	10687	814	815	5	(3 130 2447)
11503	1665	1666	3	(7 314 4418)	11527	2275	2276	5	(45 614 5275)
11551	152	11398	7	(9 642 2970)	11863	3918	7944	3	(3 504 3814)
11959	1644	1645	3	(3 554 4937)	12511	5161	7349	3	(17 706 6344)
12823	3597	3598	3	(3 502 1570)	13399	5719	5720	3	(9 182 1906)
13711	922	923	6	(7 742 1121)	13759	2615	2616	6	(21 278 5716)
13831	4021	9809	6	(33 248 9406)	13999	6235	7763	3	(7 416 187)
14551	3431	11119	3	(7 416 4461)	14983	458	14524	3	(13 734 7066)
15511	466	467	3	(13 1228 144)	15559	846	14712	3	(5 1012 2236)
15679	2099	2100	11	(15 880 1675)	15727	803	14923	3	(21 286 11236)
15823	5892	9930	3	(25 1272 4265)	16111	5328	10782	7	(3 1196 15989)
16183	6771	9411	3	(3 438 4135)	16231	7810	8420	3	(21 970 4058)
16447	7524	8922	3	(5 1150 10010)	16567	5222	11344	3	(47 482 8851)
17047	2720	14326	3	(17 1394 14898)	17191	2923	14267	3	(9 154 9778)
19087	1753	17333	19	(23 168 13024)	19543	9009	9010	3	(29 1218 5178)
19759	4057	4058	3	(27 796 15047)	20023	1209	1210	3	(31 1178 16355)
20071	2462	2463	3	(77 582 15614)	20431	6858	6859	3	(15 1404 5788)
20719	7722	7723	3	(25 134 11714)	21559	8653	12905	15	(37 382 18873)
21799	1784	1785	7	(3 244 7141)	22063	10835	10836	5	(9 640 18662)
22303	10310	10311	6	(7 1476 10704)	23671	9253	9254	3	(49 762 16093)
23719	635	636	6	(3 274 1910)	23767	2271	21495	3	(3 1078 9089)
24103	7272	7273	5	(5 1228 851)	24151	10319	10320	6	(3 1390 17130)
24223	11578	12644	3	(11 1576 8223)	24439	1001	23437	6	(63 656 7981)
24631	8158	16472	3	(9 1660 21981)	24847	4464	4465	3	(7 642 9598)
24919	9567	9568	3	(51 150 4853)	25183	10770	14412	5	(7 762 22649)
25303	2454	2455	3	(15 518 4241)	25423	3486	21936	3	(37 874 19814)
25447	11430	11431	3	(29 940 14215)	25999	2424	23574	7	(29 796 9557)
26407	2916	2917	5	(7 1866 3882)	27271	4618	22652	6	(21 446 7066)
27487	7221	7222	3	(15 1228 8364)	27943	1577	1578	5	(11 1622 9414)
28279	4639	23639	3	(5 1396 7972)	28447	8615	8616	3	(7 1938 9877)
28687	8778	8779	6	(3 1484 26339)	29023	4549	24473	3	(9 298 8131)
29191	5804	5805	7	(31 806 18469)	29311	7441	21869	3	(27 182 19210)
29983	10517	10518	5	(37 2468 26167)	30103	13060	13061	3	(45 770 8530)
30559	11698	18860	7	(15 480 3053)	30703	13123	17579	3	(35 230 4720)
31039	11534	11535	7	(5 1418 13436)	31183	3203	3204	3	(19 816 29320)
31231	13410	13411	6	(3 844 9004)	31327	177	31149	6	(35 1364 18712)
32719	14059	14060	3	(7 1520 4521)	32839	5305	27533	6	(49 974 13030)
33151	2536	2537	3	(25 1224 3672)	33223	682	32540	10	(19 260 28834)
33247	9084	9085	5	(23 1938 4194)	33391	4744	28646	6	(5 1416 32871)
33751	9606	9607	6	(21 1552 25947)	33871	5564	5565	15	(11 1984 3624)
34039	2341	2342	3	(5 1322 15151)	34303	14547	14548	17	(5 264 16545)
34351	16235	18115	3	(5 2678 15570)	34687	5954	28732	5	(17 1576 27744)
34807	15449	15450	3	(41 1036 31904)	35023	16870	16871	5	(21 2628 9599)

Table III (2): Data for Lemma 3.5 with  $t = 3$ 

$p$	$\sqrt{2}$	$\xi$	$\omega$	$(x_1 \ x_2 \ b_l)$	$p$	$\sqrt{2}$	$\xi$	$\omega$	$(x_1 \ x_2 \ b_l)$
35407	16249	16250	6	(27 424 3829)	36151	16215	19935	3	(5 2882 15185)
36319	16163	20155	6	(7 2194 34978)	36559	9369	27189	6	(23 796 35597)
36583	11617	11618	7	(17 1204 20531)	37039	9029	9030	3	(15 2336 32993)
37159	1647	1648	3	(83 2772 27397)	37831	14322	23508	3	(19 68 10563)
38239	2458	35780	13	(17 2978 13664)	38767	6548	32218	5	(7 1706 10589)
38791	12996	12997	6	(13 2372 35560)	39079	14236	14237	3	(3 334 17870)
39343	6851	6852	3	(13 1600 2469)	39367	15248	24118	3	(17 2296 32453)
39631	12325	12326	3	(13 1630 10757)	39679	14608	25070	6	(25 3106 33390)
39727	15337	24389	3	(45 1650 37110)	40039	7049	32989	6	(49 2048 28444)
40519	14617	25901	6	(5 764 8736)	40591	18519	22071	13	(9 2458 790)
40639	14583	26055	7	(7 66 28708)	40759	5421	35337	3	(7 1944 33883)
41047	9099	9100	5	(31 184 8972)	41263	20066	21196	5	(11 294 20539)
41887	3281	38605	3	(25 1286 35439)	42391	3578	3579	6	(11 174 34699)
42727	18834	18835	3	(7 2334 19611)	43159	14978	28180	3	(21 2192 38277)
43207	12625	30581	3	(35 2052 27423)	43591	3456	40134	11	(9 2138 12312)
43783	17454	17455	3	(29 2992 5942)	43951	13414	30536	6	(9 2548 19517)
44263	11318	32944	3	(7 50 19121)	44839	10915	33923	6	(7 3694 43562)
45007	3401	41605	3	(27 660 13916)	45343	8647	36695	6	(3 1712 34593)
45439	9490	9491	3	(5 874 8073)	45943	802	803	6	(7 3466 4523)
46279	14370	31908	3	(11 2356 36401)	46327	18163	28163	3	(23 300 41500)
46399	9441	9442	3	(25 1400 29004)	46591	20875	25715	6	(57 1118 38780)
47119	895	896	3	(39 1258 3622)	47143	2755	44387	6	(37 3068 26110)
47287	19519	19520	5	(13 1914 872)	47623	21963	21964	3	(7 2798 16464)
47743	12607	12608	3	(17 1404 31464)	47911	9626	38284	3	(7 242 43710)
48463	2660	45802	5	(7 92 28591)	48871	6182	6183	3	(27 1378 10974)
49639	20421	29217	3	(13 1952 28716)	49783	7497	42285	3	(31 2758 11256)
49831	4049	45781	12	(11 2910 46233)					

Table VI: Data for Lemma 3.5 with  $t \geq 7$ 

$p$	$\sqrt{2}$	$\xi$	$\omega$	$t$	$(x_1 \ x_2 \ b_l)$	$(y \ z \ u \ v)$
4943	696	4246	7	7	(53 82 3166)	(343 3198 1461 3)
8527	3253	5273	5	7	(81 146 710)	(125 7876 5688 168)
10039	4473	5565	3	7	(17 132 8360)	(27 1140 4875 5467)
10151	2010	2011	7	7	(275 348 1964)	(343 8111 10034 9430)
10847	4895	5951	5	11	(91 224 2763)	(125 2580 9241 6049)
12391	1008	11382	26	7	(187 378 2800)	(5185 8104 6516 3353)
13903	755	13147	3	7	(49 124 5002)	(27 10643 9958 10427)
18439	7031	11407	3	7	(147 340 16570)	(27 7946 11021 12865)
19447	4954	14492	3	7	(69 446 912)	(27 5589 6227 10849)
19727	2069	17657	5	7	(95 678 7086)	(125 4 5225 16676)
22111	8135	8136	6	11	(171 272 598)	(216 6579 10968 21534)
22727	11002	11003	5	11	(253 384 1456)	(125 16677 19820 14537)
22751	4520	4521	21	7	(133 376 3084)	(9261 21751 14040 6591)
22807	967	21839	3	7	(17 308 15313)	(27 4483 13920 12189)
23447	11043	12403	5	19	(263 284 915)	(125 12775 18811 1638)
26111	4481	4482	7	7	(163 912 2150)	(343 2383 23740 24916)
27791	3994	3995	7	7	(125 636 25546)	(343 26972 22113 13861)
29863	6340	23522	10	7	(51 1058 24203)	(1000 4365 15003 23904)
30367	11150	11151	5	7	(87 872 18453)	(125 2355 9015 18220)
32159	10020	22138	7	7	(127 528 5573)	(343 21163 12410 15596)
34591	6359	28231	3	15	(89 136 18522)	(27 14997 33228 101)
36871	10031	26839	15	15	(101 370 21515)	(3375 31446 5234 1813)
36919	10227	26691	3	7	(113 564 34736)	(27 6853 33992 20623)
37423	1128	1129	6	7	(5 762 22589)	(216 23199 33164 23369)
37591	9526	9527	6	15	(393 600 26042)	(216 6199 30753 34540)
39383	11060	11061	5	7	(39 1244 32621)	(125 2365 25033 18445)
41903	7749	7750	5	7	(85 544 25750)	(125 9674 23059 23700)
42463	13088	13089	3	7	(87 130 3892)	(27 39615 10781 2891)
45631	1682	43948	12	15	(331 606 13836)	(1728 24726 25517 6226)
46831	8745	8746	3	7	(33 1314 37917)	(27 33348 30207 18584)